AN APPROXIMATE STRESS ENERGY TENSOR FOR GRAVITATIONAL FIELDS*

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ABSTRACT

An invariant formulation in Minkowski space-time of an approximation to the Einstein theory of gravitation is given. In this formulation a tensor is introduced which may be interpreted as the approximate stress energy tensor of the gravitational field. Conservation laws involving this tensor and the material stress energy tensor are formulated. The behavior of these tensors under "gauge transformations" of the weak gravitational fields is discussed. The classical limit of the conservation of energy equation is studied and the results are compared to some observations of H. Bondi on a possible analogue of the Poynting vector for a gravitational field.

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1. Introduction

It is the main purpose of this paper to formulate and discuss conservation laws in invariant form in Minkowski space-time for an approximate version of the Einstein theory of gravitation. These laws will involve the approximate energy and momentum of the material and gravitational fields. The discussion will be mainly concerned with a first approximation to the Einstein theory but may be extended to higher approximations. We shall relate the results obtained to some observations of H. Bondi[(1)] concerning an analogue to the Poynting vector for classical time-dependent gravitational fields.

The Minkowski space-time will be used as the underlying space in which the discussion will take place. In principle any fixed Riemonnian space-time may be used. There are, however, two reasons for choosing the Minkowski one: (a) with this choice the Newtonian approximation is readily obtained from the first approximation given below by neglecting terms of the order of $1/c^2$ and (b) the underlying space-time admits a ten parameter group of motions, the inhomogeneous Lorentz group. Use is made of the latter fact in formulating conserved quantities.

The approximate theory mentioned above is obtained by considering the metric tensor $g_{\mu\nu}$ of space-time as defined over the Minkowski space by a convergent power series expansion in

$$k = \frac{8\pi G}{c^2} = 1.864 \times 10^{-27} \text{ cm gr}^{-1} \tag{1.1}$$

where G is Newton's constant of gravitation and c is the velocity of light of the special theory of relativity.

We assume that

$$g_{\mu\nu} = \eta_{\mu\nu} + k\, h_{\mu\nu} + \frac{k^2}{2} h_{(2)\mu\nu} + \cdots \tag{1.2}$$

where $\eta_{\mu\nu}$ is the metric tensor of the Minkowski space-time. The coordinate

system in which equation (1.2) holds may be an arbitrary one. The equations satisfied by the $g_{\mu\nu}$, that is $h_{\mu\nu}$, $h_{(2)\mu\nu}$ ... will be derived from the Einstein field equations:

$$G_{\mu\nu} = -kc^2 T_{\mu\nu} \tag{1.3}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R , \tag{1.4}$$

$R_{\mu\nu}$ is the Ricci tensor and R the scalor curvature tensor formed from the $g_{\mu\nu}$. The tensor $T_{\mu\nu}$ is the stress energy tensor of the matter "creating" the gravitational field.

Both the tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ may be considered as functions of k and written as

$$G_{\mu\nu} = G_{(0)\mu\nu} + k\, G_{(1)\mu\nu} + \frac{k^2}{2} G_{(2)\mu\nu} + \cdots\cdot \tag{1.5}$$

$$T_{\mu\nu} = T_{(0)\mu\nu} + k\, T_{(1)\mu\nu} + \frac{k^2}{2} T_{(2)\mu\nu} + \cdots\cdot \tag{1.6}$$

It is evident from equation (1.2) that

$$G_{(0)\mu\nu} \equiv 0 \tag{1.7}$$

The following discussion will center about the discussion of the equations

$$\left[(G_{\mu\nu} + kC^2\, T_{\mu\nu})\, g^{\nu\rho}\right]_{;\rho} = 0 \tag{1.8}$$

which are consequences of equations (1.3). In equation (1.8) the semi-colon denotes the covariant derivative with respect to the metric tensor $g_{\mu\nu}$. Because of the Bianchi identities, we have

$$(G_{\mu\nu}\, g^{\nu\rho})_{;\rho} \equiv 0 . \tag{1.9}$$

If equations (1.5) and 1.6) are substituted into equations (1.3) and the resulting equations are regarded as identities in k we obtain

$$G_{(n)\mu\nu} = -\, nc^2\, T_{(n-1)\mu\nu} . \tag{1.10}$$

These equations may be regarded as differential equations for the determination

of the $h_{(n)\mu\nu}$ in terms of $h_{(m)\mu\nu}$ and $T_{(m)\mu\nu}(m = 1,2, \ldots n-1)$. The $T_{(m)\mu\nu}$ must be such that

$$\left[\frac{d^m}{dk^m}\left(kc^2\, T_{\mu\nu}\, g^{\nu\rho}\right)_{;\rho}\right]_{k=o} = 0 \tag{1.11}$$

where $T_{\mu\nu}$ is given by equation (1.6), and the $g^{\nu\rho}$ are functions of k which satisfy

$$\frac{dg^{\nu\rho}}{dk} = -g^{\nu\sigma}\,\frac{dg_{\sigma\tau}}{dk}\,g^{\tau\rho} \tag{1.12}$$

We also require that

$$\left(T_{(0)\mu\nu}\,\eta^{\mu\rho}\right)_{,\rho} = 0 \tag{1.13}$$

where the comma denotes the covariant derivative with respect to the tensor $\eta_{\mu\nu}$.

Equations (1.12) and (1.13) are equations for the determination of the $T_{(n)\mu\nu}$ in terms of $h_{(r)\mu\nu}$ and $T_{(s)\mu\nu}$ with r = 1,2, ... n and s = 0,2, ...n-1.

2. <u>The Calculation of $G^{\mu}_{(1)\nu}$</u>

We begin our discussion by considering the expansion of the Christoffel symbols as power series in k. Thus

$$\left\{{\mu \atop \nu\ \sigma}\right\} = \frac{1}{2}\, g^{\mu\rho}\left(g_{\nu\rho|\sigma} + g_{\rho\sigma|\nu} - g_{\nu\sigma|\rho}\right) \tag{2.1}$$

where we have used the notation

$$\frac{\partial g_{\nu\rho}}{\partial x^{\sigma}} = g_{\nu\rho|\sigma} \tag{2.2}$$

We may write

$$\left\{{\mu \atop \nu\ \sigma}\right\} = \left\{{\mu \atop \nu\ \sigma}\right\}_{(0)} + k\left\{{\mu \atop \nu\ \sigma}\right\}_{(1)} + \frac{k^2}{2!}\left\{{\mu \atop \nu\ \sigma}\right\}_{(2)} + \cdots$$

where

$$\left\{{\mu \atop \nu\ \sigma}\right\}_{(n)} = \left(\frac{d^n}{dk^n}\left\{{\mu \atop \nu\ \sigma}\right\}\right)_{k=o} \tag{2.3}$$

Thus

$$\left\{ {\mu \atop \nu\ \sigma} \right\}_{(o)} = \frac{1}{2}\,\eta^{\mu\rho}(\eta_{\nu\rho|\sigma} + \eta_{\rho\sigma|\nu} - \eta_{\sigma\nu|\rho}) , \tag{2.4}$$

the Christoffel symbol calculated from the η's. In a galilean coordinate system

$$\left\{ {\mu \atop \nu\ \sigma} \right\}_{(o)} = 0 \tag{2.5}$$

It may be verified that

$$A^{\mu}_{\nu\ \sigma} = \left\{ {\mu \atop \nu\ \sigma} \right\}_{(1)} = \frac{1}{2}\,\eta^{\mu\rho}(h_{\nu\rho,\sigma} + h_{\rho\sigma,\nu} - h_{\sigma\nu,\rho}) \tag{2.6}$$

where, as above the comma denotes the covariant derivative with respect to the tensor $\eta_{\mu\nu}$. It follows from equation (2.6) that

$$\left\{ {\mu \atop \nu\ \mu} \right\}_{(1)} = \frac{1}{2}\,\eta^{\mu\rho}\ h_{\mu\rho,\nu} = \frac{1}{2}\,h_{,\nu} \tag{2.7}$$

where

$$h = \eta^{\mu\rho}\ h_{\mu\rho} \tag{2.8}$$

The Ricci tensor is defined by the equation

$$R_{\mu\nu} = -\left\{ {\sigma \atop \mu\ \nu} \right\}_{|\sigma} + \left\{ {\sigma \atop \mu\ \sigma} \right\}_{|\nu} - \left\{ {\rho \atop \mu\ \nu} \right\}\left\{ {\sigma \atop \rho\ \sigma} \right\} + \left\{ {\tau \atop \mu\ \sigma} \right\}\left\{ {\sigma \atop \tau\ \nu} \right\} \tag{2.9}$$

Hence

$$R_{(0)\mu\nu} = 0$$

and

$$R_{(1)\mu\nu} = -\left\{ {\sigma \atop \mu\ \nu} \right\}_{(1),\sigma} + \left\{ {\sigma \atop \mu\ \sigma} \right\}_{(1),\nu} \tag{2.10}$$

On substituting from equation (2.6) and (2.7) into equation (2.10) we obtain

$$R_{(1)\mu\nu} = -\frac{1}{2}\,\eta^{\rho\sigma}(h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho} - \eta_{\rho\nu}\ h_{,\mu})_{,\sigma} \tag{2.11}$$

Since $R_{(0)\mu\nu} = 0$ we have

$$R_{(1)} = \eta^{\mu\nu}\ R_{(1)\mu\nu} = -\,\eta^{\rho\sigma}(h_{\rho\alpha,\beta}\ \eta^{\alpha\beta} - h_{,\rho})_{,\sigma} \tag{2.12}$$

Thus we may write

$$G_{(1)\mu\nu} = R_{(1)\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R_{(1)}$$

$$= -\frac{1}{2}\eta^{\sigma\rho}(k_{\rho\nu,\mu} + k_{\mu\rho,\nu} - k_{\mu\nu,\rho} - \eta_{\mu\nu} k_{\rho\alpha,\beta}\eta^{\alpha\beta})_{,\sigma}$$

$$= -\frac{1}{2}\eta^{\rho\sigma}(k_{\rho\nu,\mu} - k_{\mu\nu,\rho} - \eta_{\mu\nu}k_{\rho\alpha,\beta}\eta^{\alpha\beta} + \eta_{\rho\nu} k_{\mu\alpha,\beta}\eta^{\alpha\beta})_{,\sigma} \tag{2.13}$$

where

$$k_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{2.14}$$

Because the Minkowski space is a flat space it follows that

$$G_{(1)}{}^{\mu}{}_{\nu} = \eta^{\tau\mu} G_{\tau\nu} \tag{2.15}$$

and the order of covairant differentiation is immaterial, that is

$$t \ldots{}_{,\alpha\beta} = t \ldots{}_{,\beta\alpha} \,.$$

It may be verified by using equations (2.14) and (2.15) that

$$G_{(1)}{}^{\mu}{}_{\nu,\mu} = 0 \tag{2.16}$$

3. The Calculation of $G^{\mu\nu}_{(2)}$

In this section we shall evaluate the above tensor in terms of $h_{\mu\nu}$, $h_{(2)\mu\nu}$ and their first and second derivatives. We shall show that it may be written as a sum of two tensors. One of these contains the second derivatives of $h_{\mu\nu}$ and $h_{(2)\mu\nu}$ and has a vanishing divergence. The other is a function of the $h_{\mu\nu}$ its first derivatives and $G_{(1)\mu\nu}$.

It follows from equations (2.1) that

$$\left\{ {\mu \atop \nu\ \sigma} \right\}_{(2)} = B_{\nu}{}^{\mu}{}_{\sigma} - 2h^{\mu}_{\rho} A^{\rho}_{\nu\sigma} \tag{3.1}$$

where

$$B^{\mu}_{\nu\sigma} = \eta^{\mu\rho}\frac{1}{2}(h_{(2)\nu\rho,\sigma} + h_{(2)\sigma\rho,\nu} - h_{(2)\nu\sigma,\rho}) \tag{3.2}$$

and $A^{\rho}_{\nu\sigma}$ is defined by equations (2.6).

By differentiating equation (2.9) twice with respect to k and setting k = 0 we obtain

$$R_{(2)\sigma\tau} = - \left\{ {\rho \atop \sigma\ \tau} \right\}_{(2),\rho} + \left\{ {\rho \atop \sigma\ \rho} \right\}_{(2),\tau} - 2A^{\rho}_{\sigma\tau} A^{\lambda}_{\rho\lambda} + 2A^{\rho}_{\sigma\lambda} A^{\lambda}_{\rho\tau} \tag{3.3}$$

Since

$$G^{\mu\nu} = (g^{\mu\sigma} g^{\nu\tau} - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau}) R_{\sigma\tau} ,$$

it follows that

$$\begin{aligned} G^{\mu\nu}_{(2)} &= (\eta^{\mu\sigma} \eta^{\nu\tau} - \frac{1}{2} \eta^{\mu\nu} \eta^{\sigma\tau}) R_{(2)\sigma\tau} \\ &- 2(h^{\mu\sigma}\eta^{\nu\tau} + h^{\nu\tau} \eta^{\mu\sigma} - \frac{1}{2} h^{\mu\nu} \eta^{\sigma\tau} - \frac{1}{2} h^{\sigma\tau}\eta^{\mu\nu})R_{(1)\sigma\tau} \end{aligned} \tag{3.4}$$

In view of equations (2.10) and (3.3) we may write the above equation as

$$\begin{aligned} G^{\mu\nu}_{(2)} = &\Bigg[\left(\eta^{\mu\sigma} \eta^{\nu\tau} - \frac{1}{2} \eta^{\mu\nu}\eta^{\sigma\tau}\right)\left(- \left\{ {\alpha \atop \sigma\ \tau} \right\}_{(2)} + \left\{ {\rho \atop \sigma\ \rho} \right\}_{(2)} \delta^{\alpha}_{\tau}\right) \\ &+ 2\left(h^{\mu\sigma}\eta^{\nu\tau} + h^{\nu\tau} \eta^{\mu\sigma} - \frac{1}{2} h^{\mu\nu}\eta^{\sigma\tau} - \frac{1}{2} h^{\sigma\tau}\eta^{\mu\nu}\right)\left(A^{\alpha}_{\sigma\tau} - A^{\rho}_{\sigma\rho} \delta^{\alpha}_{\tau}\right)\Bigg]_{,\alpha} \\ &+ 2\left(\eta^{\mu\sigma}\eta^{\nu\tau} - \frac{1}{2} \eta^{\mu\nu} \eta^{\sigma\tau}\right)\left(A^{\rho}_{\sigma\lambda} A^{\lambda}_{\rho\tau} - A^{\rho}_{\sigma\tau} A^{\lambda}_{\rho\lambda}\right) \\ &- 2\left(h^{\mu\sigma}\eta^{\nu\tau} + h^{\nu\tau} \eta^{\mu\sigma} - \frac{1}{2} h^{\mu\nu} \eta^{\sigma\tau} - \frac{1}{2} h^{\sigma\tau} \eta^{\mu\nu}\right)_{,\alpha}\left(A^{\alpha}_{\sigma\tau} - A^{\rho}_{\sigma\rho} \delta^{\alpha}_{\tau}\right) \end{aligned}$$

It may be verified that

$$G^{\mu\nu}_{(2)} = H^{\mu\nu} + \frac{2c^2}{k} E^{\mu\nu} \tag{3.5}$$

where

$$H^{\mu\ \nu}_{\ ,\ \nu} = 0 \tag{3.6}$$

and

$$
\begin{aligned}
H^{\mu\nu} = \Big[& \Big(\eta^{\mu\sigma}\,\eta^{\nu\tau} - \tfrac{1}{2}\,\eta^{\mu\nu}\,\eta^{\sigma\tau}\Big)\Big(-B^{\alpha}_{\sigma\tau} + \delta^{\alpha}_{\tau}\,B^{\rho}_{\sigma\rho}\Big) \\
& + \Big(\ell^{\mu\alpha} - \tfrac{1}{4}\,\ell\eta^{\mu\alpha}\Big)_{,\tau}\eta^{\tau\nu} + \Big(\ell^{\nu\alpha} - \tfrac{1}{4}\,\ell\eta^{\nu\alpha}\Big)_{,\tau}\eta^{\mu\tau} - \Big(\ell^{\mu\nu} - \tfrac{1}{4}\,\ell\eta^{\mu\nu}\Big)_{,\tau}\eta^{\tau\alpha} \\
& - \eta^{\mu\nu}\Big(\ell^{\alpha\tau} - \tfrac{1}{4}\,\ell\eta^{\alpha\tau}\Big)_{,\tau} + h^{\nu\alpha}_{\ ,\tau}\,h^{\tau\mu} - h^{\mu\nu}\,h^{\alpha\tau}_{\ ,\tau} \\
& + h^{\mu\alpha}_{\ ,\tau}\,h^{\tau\nu} - h^{\mu\nu}_{\ ,\tau}\,h^{\alpha\tau}\Big]_{,\alpha} - h^{\mu\tau}_{\ ,\alpha}\,h^{\alpha\nu}_{\ ,\tau} + h^{\mu\alpha}_{\ ,\alpha}\,h^{\nu\tau}_{\ ,\tau} \\
& - \Big[h\Big(h^{\rho\mu}\eta^{\alpha\nu} + h^{\alpha\nu}\eta^{\rho\mu} - h^{\mu\nu}\eta^{\alpha\rho} - \eta^{\mu\nu}\,h^{\rho\alpha}\Big)\Big]_{,\rho\alpha} \\
& + \Big[h\,h_{,\rho}\Big(\eta^{\rho\mu}\,\eta^{\alpha\nu} - \eta^{\mu\nu}\eta^{\rho\alpha}\Big)\Big]_{,\alpha}
\end{aligned}
\tag{3.7}
$$

with

$$
\ell^{\mu\nu} = h^{\mu}{}_{\rho}h^{\rho\nu} \; . \tag{3.8}
$$

$$
\begin{aligned}
\frac{2c^2}{k}\,E^{\mu\nu} = {} & 2(\eta^{\mu\sigma}\eta^{\nu\tau} - \tfrac{1}{2}\,\eta^{\mu\nu}\eta^{\sigma\tau})(A^{\rho}_{\sigma\lambda}\,A^{\lambda}_{\rho\tau} - A^{\rho}_{\sigma\tau}\,A^{\lambda}_{\tau\lambda}) \\
& - 2(h^{\mu\sigma}\eta^{\nu\tau} + h^{\nu\tau}\eta^{\mu\sigma} - \tfrac{1}{2}\,h^{\mu\nu}\eta^{\sigma\tau} - \tfrac{1}{2}\,h^{\sigma\tau}\eta^{\mu\nu})_{,\alpha}\,(A^{\alpha}_{\sigma\tau} - A^{\rho}_{\sigma\rho}\,\delta^{\alpha}_{\tau}) \\
& + \Big[h(k^{\rho\mu}\,\eta^{\alpha\nu} + k^{\alpha\nu}\,\eta^{\rho\mu} - k^{\mu\nu}\,\eta^{\alpha\rho} - \eta^{\mu\nu}\,k^{\rho\alpha})_{,\rho}\Big]_{,\alpha} \\
& + h^{\mu\tau}_{\ ,\alpha}\,h^{\alpha\nu}_{\ ,\tau} - h^{\mu\alpha}_{\ ,\alpha}\,h^{\nu\tau}_{\ ,\tau}
\end{aligned}
\tag{3.9}
$$

where as before

$$
k^{\mu\nu} = h^{\mu\nu} - \tfrac{1}{2}\,\eta^{\mu\nu}\,h \; .
$$

Equation (3.9) may be shown to be equivalent to

$$
\begin{aligned}
\frac{2c^2}{k}\,E^{\mu\nu} = {} & -k^{\mu}_{\tau,\rho}\,k^{\tau\rho}_{\ ,\sigma}\,\eta^{\sigma\nu} - k^{\nu}_{\tau,\rho}\,k^{\tau\rho}_{\ ,\sigma}\,\eta^{\sigma\mu} \\
& + k^{\mu}_{\rho,\tau}\,k^{\nu\rho}_{\ ,\lambda}\,\eta^{\tau\lambda} + k^{\rho\tau}_{\ ,\rho}\,k^{\mu\nu}_{\ ,\tau} \\
& - k^{\mu\rho}_{\ ,\rho}\,k^{\nu\lambda}_{\ ,\lambda} + \tfrac{1}{2}\,k^{\tau\rho}_{\ ,\sigma}\,k_{\tau\rho,\lambda}\eta^{\sigma\mu}\eta^{\lambda\nu} - \tfrac{1}{4}\,h_{,\rho}h_{,\sigma}\,\eta^{\rho\mu}\eta^{\sigma\nu} - 2h\,G^{\mu\nu}_{(1)} \\
& + \tfrac{1}{2}\,\eta^{\mu\nu}\Big[k_{\alpha\sigma,\tau}\,k^{\alpha\tau}_{\ ,\rho}\,\eta^{\rho\sigma} - \tfrac{1}{2}\,k_{\alpha\sigma,\tau}\,k^{\alpha\sigma}_{\ ,\rho}\,\eta^{\rho\tau} + \tfrac{1}{4}\,h_{,\rho}h_{,\sigma}\eta^{\rho\sigma}\Big]
\end{aligned}
\tag{3.10}
$$

It follows from equations (3.5) and (3.6) that

$$G^{\mu\nu}_{(2),\nu} = \frac{2c^2}{k} E^{\mu\nu}_{,\nu} \tag{3.11}$$

In view of the field equations, that is equations (1.10), and equation (2.16) this equation may be written as

$$\left(T^{\mu\nu}_{o} + kT^{\mu\nu}_{(1)} + E^{\mu\nu}\right)_{,\nu} = 0 \tag{3.12}$$

Equation (3.11) holds for arbitrary $h_{\mu\nu}$ and $h_{(2)\mu\nu}$, that is it is an identity in these quantities.

Equations (3.12) are the approximate equations of motion of the matter represented by the tensor $T^{\mu\nu}$ (cf equation (1.6)), correct to terms involving k^2. They are written in invariant form in the Minkowski space. If $h_{\mu\nu}$ is interpreted as a tensor in this space (approximately) representing the gravitational field created by the matter, then we may regard the last term in equations (3.12) as the "stress-energy" tensor of the gravitational field.

It is the object of this paper to evaluate the right hand side of equations (3.9) for a particular choice of the tensor $T_{(0)\mu\nu}$, that is, for a particular choice of $h_{\mu\nu}$. We first make some remarks concerning some consequences of these equations.

4. Conservation Equations

We may write equations (3.12) as

$$(M^{\mu\nu} + E^{\mu\nu})_{,\mu} = 0 \tag{4.1}$$

where

$$M^{\mu\sigma} = T^{\mu\sigma}_{(0)} + k\, T^{\mu\sigma}_{(1)} \tag{4.2}$$

Multiplying equation (4.1) by an arbitrary vector field of Minkowski space-time, λ_ν and summing we obtain

$$\left(\lambda_\nu(M^{\mu\nu} + E^{\mu\nu})\right)_{,\mu} - \frac{1}{2}(M^{\mu\nu} + E^{\mu\nu})(\lambda_{\mu,\nu} + \lambda_{\nu,\mu}) = 0 \ ,$$

since both $M^{\mu\nu}$ and $E^{\mu\nu}$ are symmetric.

When λ_μ is a Killing vector of Minkowski space-time, that is when

$$\lambda_{\mu,\nu} + \lambda_{\nu,\mu} = 0 \tag{4.3}$$

we obtain the conservation equation

$$\left(\lambda_\nu \, (M^{\mu\nu} + E^{\mu\nu})\right)_{,\mu} = 0 \ . \tag{4.4}$$

As is well known the general solution of equations (4.3) is given by

$$\lambda_\mu = F_{\mu\nu} \, X^\nu + a_\mu \tag{4.5}$$

in a galilean coordinate system in Minkowski space-time where

$$F_{\mu\nu} = - F_{\nu\mu}$$

and are independent of x^ρ and the a_μ are constants. There are thus ten linearly independent λ_μ and associated with each of these there is a conservation theorem of the form of equation (4.4).

The four vectors which in a galilean coordinate system have the coordinate

$$\lambda^{(\alpha)}_\mu = \eta_{\alpha\mu} \qquad\qquad \alpha = 1,2,3,4 \tag{4.6}$$

will be said to be associated with the conservation of energy and momentum.

Equation (4.4) implies that

$$\int \lambda_\nu \, (M^{\mu\nu} + E^{\mu\nu}) n_\mu \, d^3v = 0 \tag{4.7}$$

where the integral is taken over a closed three dimensional hypersurface in Minkowski space-time and

$$n_\mu \, d^3 \, v = \sqrt{-\eta}\,\epsilon_{\mu\nu\sigma\tau} \, \frac{\partial x^\nu}{\partial u} \frac{\partial x^\sigma}{\partial v} \frac{\partial x^\tau}{\partial w} \, du \, dv \, dw \tag{4.8}$$

if u, v and w are variables giving a parameterization of the hypersurface.

For use in later sections we drive an equation based on the Bianchi identity:

$$G^{\mu\nu}_{\ ;\nu} = G^{\mu\nu}_{\ |\nu} + G^{\rho\nu} \left\{ {\mu \atop \rho\ \mu} \right\} + G^{\mu\rho} \left\{ {\nu \atop \mu\ \rho} \right\} = 0 \ .$$

If this equation is differentiated twice with respect to k and k is set equal to zero we obtain

$$G^{\mu\nu}_{(2),\nu} = -2\, G^{\rho\mu}_{(1)} \left\{ {\nu \atop \rho\ \nu} \right\}_{(1)} - 2G^{\nu\rho}_{(1)} \left\{ {\mu \atop \nu\ \rho} \right\}_{(1)}$$

That is

$$G_{(2)\mu}{}^{\nu}{}_{,\nu} = G_{(1)}^{\nu\rho}\, h_{\nu\rho,\mu} - h_{,\nu}\, G^{\nu}_{(1)\mu} - 2G^{\nu\rho}_{(1)}\, h_{\nu\mu,\rho}$$

In view of equations (1.10), (2.16) and (4.2) this equation may be written as

$$M^{\nu}_{\mu,\nu} = \frac{k}{2} \left[T^{\nu\rho}_{(0)} h_{\nu\rho,\mu} - h_{,\nu}\, T^{\nu}_{(0)\mu} - 2\, T^{\nu\rho}_{(0)} h_{\rho\mu,\nu} \right] \tag{4.9}$$

5. Gauge Invariance

In this section we shall discuss the effects of a transformation of coordinates in the Reimannian space-time on the tensors $h_{\mu\nu}$ and $h_{(2)\mu\nu}$ in the Minkowski space-time. We recall that in any coordinate system

$$h_{(n)\mu\nu} = \left(\frac{d^n}{dk^n}\, g_{\mu\nu} \right)_{k=o} . \tag{5.1}$$

Under the transformation of coordinates in the Reimannian space-time defined by the equations

$$y^{\mu} = y^{\mu}(x) \tag{5.2}$$

the tensor $g_{\mu\nu}$ transforms as

$$g_{\mu\nu}(x) = g^{*}_{\sigma\tau}(y(x))\; y^{\sigma}_{|\mu}\; y^{\tau}_{|\nu} \tag{5.3}$$

It then follows that if the functions y^{μ} are independent of k that the quantities $h_{(n)\mu\nu}$ transforms as tensors in the Minkowski space-time under the transformation given by equations (5.2) where these are now interpreted as a transformation of coordinates in the Minkowski space-time.

If the functions y^μ depend on k, it follows from equations (5.2) and (5.3) that

$$\overset{*}{h}_{(n)\mu\nu} \equiv \left(\frac{d^n}{dk^n}\, \overset{*}{g}_{\mu\nu}\right)_{k=o}$$

does not transform as a tensor in the Minkowski space-time. In this case equations (5.2) which may be written as

$$y^\mu = y^\mu(x;k) \tag{5.4}$$

may be interpreted either as a transformation of coordinates for fixed k or as a congruence of curves for variable k.

It is sufficient to discuss the case where equations (5.4) are such that

$$\left(y^\mu_{\rceil\nu}\right)_{k=o} = \delta^\mu_\nu \tag{5.5}$$

That is,

$$y^\mu = x^\mu + kf^\mu(x;k) \tag{5.6}$$

For a general transformation of the form of (5.4) is obtained from (5.6) by following it by a transformation independent of k. Let us write

$$\left(\frac{d^n y^\mu}{dk^n}\right)_{k=o} = a^\mu_{(n)} \tag{5.7}$$

and set

$$a^\mu_{(1)} = a^\mu \ . \tag{5.8}$$

The functions a^μ are the components of a contravariant vector field, the vector field tangent to the congruence of curves (5.4) at the point x^μ. In fact under the transformation of coordinates

$$x'^\sigma = g^\sigma(x)$$

with the definition

$$y'^\sigma = g^\sigma(y)$$

we have

$$a'^{\sigma} = \left(\frac{dy'^{\sigma}}{dk}\right)_{k=o} = \left(\frac{\partial g^{\sigma}}{\partial y^{\tau}}\right)_{k=o} a^{\tau} = \frac{\partial x'^{\sigma}}{\partial x^{\tau}} a^{\tau} .$$

However the functions $a^{\mu}_{(2)}(x)$ do not have a vector transformation law. Indeed we have

$$a'^{\sigma}_{(2)} = \left(\frac{d^2 y'^{\sigma}}{dk^2}\right)_{k=o} = \frac{\partial x'^{\sigma}}{\partial x^{\tau}} a^{\tau}_{(2)} + \frac{\partial^2 x'^{\sigma}}{\partial x^{\rho} \partial x^{\tau}} a^{\rho} a^{\tau} .$$

Note that the quantity

$$b^{\sigma} = a^{\sigma}_{(2)} + \left\{ {\sigma \atop \tau\ \rho} \right\} a^{\rho} a^{\tau} \tag{5.9}$$

does obey the transformation law of a vector.

It follows from equations (5.3) and the definition (5.1) that under the transformation (5.4) subject to equation (5.5)

$$\overset{*}{h}_{\mu\nu} = h_{\mu\nu} - a_{\mu,\nu} - a_{\nu,\mu} \tag{5.10}$$

$$\begin{aligned} \overset{*}{h}_{(2)\mu\nu} &= h_{(2)\mu\nu} - b_{\mu,\nu} - b_{\nu,\mu} - 2\eta_{\sigma\tau}\, a^{\sigma}_{,\mu}\, a^{\tau}_{,\nu} \\ &\quad - 2(\overset{*}{h}_{\mu\nu,\rho}\, a^{\rho} + \overset{*}{h}_{\mu\sigma}\, a^{\sigma}_{,\nu} + \overset{*}{h}_{\nu\sigma}\, a^{\sigma}_{,\mu}) \\ &= h_{(2)\mu\nu} - 2(h_{\mu\nu,\rho}\, a^{\rho} + h_{\mu\sigma}\, a^{\sigma}_{,\nu} + h_{\nu\sigma}\, a^{\sigma}_{,\mu}) \\ &\quad - (b_{\mu} - 2a_{\mu,\sigma}\, a^{\sigma})_{,\nu} - (b_{\nu} - 2a_{\nu,\sigma}\, a^{\sigma})_{,\mu} \end{aligned} \tag{5.11}$$

where the vector b^{μ} is given by equations (5.9) and we have made use of the fact that $\eta_{\sigma\tau}$ is the metric tensor of a flat space.

In case

$$h_{\mu\nu} = h_{(2)\mu\nu} = 0$$

$$\overset{*}{h}_{\mu\nu} = -(a_{\mu,\nu} + a_{\nu,\mu}) \tag{5.12}$$

$$h^{*}_{(2)\mu\nu} = -(b_\mu - 2a_{\mu,\sigma}\, a^\sigma)_{,\nu} - (b_\nu - 2a_{\nu,\sigma}\, a^\sigma)_{,\mu} \tag{5.13}$$

That is, even when the Reimannian space-time is flat but a non-galilean coordinate system is used which arises from a galilean one by a transformation of the type given by equations (5.4), the quantities $h_{\mu\nu}$ and $h_{(2)\mu\nu}$ need not vanish. However, they are of the form given by equations (5.12) and (5.13).

We shall call the transformation

$$h_{\mu\nu} \longrightarrow h^{*}_{\mu\nu}$$

$$h_{(2)\mu\nu} \longrightarrow h^{*}_{(2)\mu\nu}$$

where the $h^{*}_{\mu\nu}$ and $h^{*}_{(2)\mu\nu}$ are given by equations (5.10) and (5.11) a gauge transformation. It is the transformation induced on these tensors by the coordinate transformation (5.4). When $h^{*}_{\mu\nu}$ and $h^{*}_{(2)\mu\nu}$ are substituted into equations (2.13) and (3.4) we will obtain quantities we shall denote as $G^{\mu\nu *}_{(1)}$ and $G^{\mu\nu *}_{(2)}$. These are the coefficients of the first and second powers of k in the expansion of the tensor

$$R^{*\mu\nu} - \frac{1}{2} g^{\mu\nu *} R^{*} = G^{*\mu\nu}$$

which may be obtained from the tensor $G^{\mu\nu}$ by using the fact that $G^{*\mu\nu}$ arises from $G^{\mu\nu}$ by means of the transformation (5.4) and the transformation law

$$G^{*\mu\nu}(x) = G^{\sigma\tau}(x)\; y^{\mu}_{|\sigma}\; y^{\nu}_{|\tau} \tag{5.14}$$

It follows from this equation by setting k=o, by differentiating with respect to k and setting k=o, and by differentiating twice with respect to k and setting k=o that

$$G^{*\mu\nu}_{(0)} = G^{\mu\nu}_{(0)} = 0$$

$$G^{*\mu\nu}_{(1)} = G^{\mu\nu}_{(1)} \tag{5.15}$$

and

$$G^{\mu\nu *}_{(2)} = G^{\mu\nu}_{(2)} - 2\left[G^{\mu\nu}_{(1),\rho}\, a^\rho - G^{\mu\tau}_{(1)}\, a^{\nu}_{,\tau} - G^{\tau\nu}_{(1)}\, a^{\mu}_{,\tau}\right] \tag{5.16}$$

These equations may also be derived by substituting $h^*_{\mu\nu}$ and $h^*_{(2)\mu\nu}$ into the equations defining $G^{*\mu\nu}_{(1)}$ and $G^{*\mu\nu}_{(2)}$ as functions of these quantities. Since equations (3.10) hold identically in $h_{\mu\nu}$ and $h_{(2)\mu\nu}$ we have

$$G^{*\sigma\tau}_{(2),\tau} = \frac{2c^2}{k} E^{*\sigma\tau}_{,\tau} = \frac{2c^2}{k} E^{\sigma\tau}_{,\tau} + 2G^{\sigma\rho}_{(1)} a^{\tau}_{,\tau\rho} + 2G^{\rho\tau}_{(1)} a^{\sigma}_{,\tau\rho}$$

$$= 2\left(\frac{c^2}{k} E^{\sigma\tau} + G^{\sigma\tau} a^{\rho}_{,\rho} + G^{\tau\rho} a^{\sigma}_{,\rho}\right)_{,\tau} \tag{5.17}$$

as follows from equation (5.16) and (2.16).

It is a consequence of equations (5.15), (5.16) and (1.10) that

$$T^{\mu\nu*}_{(0)} + kT^{\mu\nu*}_{(1)} = T^{\mu\nu}_{(0)} + kT^{\mu\nu}_{(1)} - k\left[T^{\mu\nu}_{(0),\rho} a^{\rho} - T^{\mu\tau}_{(0)} a^{\nu}_{,\tau} - T^{\tau\nu}_{(0)} a^{\mu}_{,\tau}\right] \tag{5.18}$$

We now define the vector

$$\lambda^*_{\mu} = \lambda_{\mu} - k(a^{\rho} \lambda_{\mu,\rho} + \lambda_{\rho} a^{\rho}_{,\mu}) \tag{5.19}$$

where λ_{μ} is one of the Killing vectors of Minkowski space-time, that is, λ_{μ} satisfies equations (4.3). It may be verified as a consequence of equations (5.18) and (5.19) that

$$\lambda^*_{\mu} M^{*\mu\nu} = \lambda_{\mu} M^{\mu\nu} - k\left(a^{\rho}(\lambda_{\mu} T^{\mu\nu}_{(0)})_{,\rho} - \lambda_{\mu} T^{\mu\rho}_{(0)} a^{\nu}_{,\rho}\right) \tag{5.20}$$

where terms in k^2 have been neglected, and $M^{\mu\nu}$ and $M^{*\mu\nu}$ are defined by means of equations (4.2) and the corresponding equation for the starred quantities.

If we multiply equations (5.20) by $(1 - ka^{\sigma}_{,\sigma})$ we then obtain to the same accuracy

$$(1 - ka^{\sigma}_{,\sigma}) \lambda^*_{\mu} M^{\mu\nu*} = \lambda_{\mu} M^{\mu\nu} - k\left[(a^{\rho} \lambda_{\mu} T^{\mu\nu}_{(0)})_{,\rho} - \lambda_{\mu} T^{\mu\rho}_{(0)} a^{\nu}_{,\rho}\right]$$

and hence

$$\left[(1 - ka^{\sigma}_{,\sigma}) \lambda^*_{\mu} M^{\mu\nu*}\right]_{,\nu} = (\lambda_{\mu} M^{\mu\nu})_{,\nu} - k\left[a^{\nu}(\lambda_{\mu} T^{\mu\rho}_{(0)})_{,\rho}\right]_{,\nu}$$

$$= (\lambda_{\mu} M^{\mu\nu})_{,\nu} \tag{5.21}$$

The first form of equation (5.21) holds for an arbitrary vector λ_μ. The second form of this equation follows from the first form by virtue of the fact that λ_μ is a Killing vector and $\underset{(0)}{T}^{\mu\nu}$ has a vanishing divergence.

It follows from equation (5.21), by integration over a region of Minkowski space-time bounded by a closed three-dimensional hypersurface, that

$$\int (1 - ka^\sigma_{,\sigma}) \lambda^*_\mu M^{\mu\nu *} n_\nu d^3 v = \int \lambda_\mu M^{\mu\nu} n_\nu d^3 v$$

$$= - \int \lambda_\mu E^{\mu\nu} n_\nu d^3 v. \qquad (5.22)$$

It is of course a consequence of the first of equations (5.17) that

$$\int \lambda_\mu M^{\mu\nu *} n_\nu d^3 v = - \int \lambda_\mu E^{\mu\nu *} n_\nu d^3 v . \qquad (5.23)$$

The difference between the surface integral of $E^{\mu\nu *}$ and that of $E^{\mu\nu}$ is due to the fact that the hypersurface in Minkowski space-time into which the hypersurface defined by the equations

$$x^\mu = x^\mu(u, v, w)$$

transforms under the transformation defined by equations (5.4) differs from the former one. Thus we see that although the gravitational energy tensor $E^{\mu\nu}$ is not gauge invariant, the conserved quantities computed from it are related by equations (5.22) which take into account the fact that the gauge transformations arise from coordinate transformations in the Riemannian space-time.

6. Bondi's Relation

In this section we shall compare equation (4.4) with

$$\lambda_\nu = \eta_{4\nu} ,$$

where the $\eta_{\mu\nu}$ are evaluated in a galilean coordinate system, to a set of equations first derived by Bondi (1) from classical arguements. We shall derive his relations by studying the classical limit of the Einstein field

equations for weak fields for the case where $\underset{(0)}{T}^{\mu\nu}$ is the stress energy tensor of a perfect fluid. In forming the classical limit we shall neglect terms involving $1/c^2$.

McVittie (2) has given the $h_{\mu\nu}$ associated via equations (1.10) with such a $\underset{(0)}{T}^{\mu\nu}$. In the notation used above McVittie's results may be written as follows:

In a galilean coordinate system let

$$h_{\mu\nu} = -2\varphi\, \delta_\mu^4\, \delta_\nu^4 + (\varphi + \frac{2g_{(\mu)}}{c^2})\, \eta_{\mu\nu} \tag{6.1}$$

then

$$h = 2(\varphi + \frac{\psi}{c^2}) \tag{6.2}$$

where

$$\psi = \sum_{\mu=1}^{4} g_{(\mu)} \tag{6.3}$$

and

$$k_{\mu\nu} = -\, 2\varphi\, \delta_\mu^4\, \delta_\nu^4 - \frac{2}{c^2}\, \psi_{(\mu)}\, \eta_{\mu\nu} \tag{6.4}$$

with

$$2\psi_{(\mu)} = \psi - 2g_{(\mu)} \tag{6.5}$$

hence

$$\sum_{\mu=1}^{4} \psi_{(\mu)} = \psi \tag{6.6}$$

It may be verified from equations (2.13) that

$$G^{44}_{(1)} = c^2 \delta^{ij} \varphi_{,ij} + \psi_{(4),ij} \delta^{kj} + \sum_{(\ell)} \psi_{(\ell),\ell\ell}$$

$$G^{4i}_{(1)} = - c^2 \varphi_{,i4} - \psi_{(4),i4} - \psi_{(i),i4}$$

$$G^{ii}_{(1)} = c^2 \varphi_{,44} + \psi_{(4),44} - c^2 \psi_{(i),\rho\sigma} \eta^{\rho\sigma} + 2\psi_{(i),ii} - c^2\Big(\sum_{\ell} \psi_{(\ell),\ell\ell}\Big)$$

$$G^{ij}_{(1)} = c^2\Big(\psi_{(i)} + \psi_{(j)}\Big)_{,ij} \qquad i \neq j$$

The tensor $T^{\mu\nu}_{(0)}$ may now be calculated from Equation (1.10) where the above quantities are used for $G^{\mu\nu}_{(1)}$. If in the resulting equations we neglect the terms in $1/c^2$ we obtain for the classical limit

$$T^{44}_{(0)} = - \delta^{ij} \varphi_{,ij} = \rho$$

$$T^{4i}_{(0)} = \varphi_{,i4} = \rho U_i \tag{6.7}$$

$$T^{ii}_{(0)} = - \varphi_{,44} - 2\psi_{(i),ii} + \psi_{(i),k\ell} \delta^{k\ell} + \sum_{(\ell)} \psi_{(\ell),\ell\ell} = \rho U_i^2 + p$$

$$T^{ij}_{(0)} = - (\psi_{(i)} + \psi_{ij})_{,ij} = \rho U_i U_j$$

The extreme right hand sides of equations (6.7) are obtained from the classical limit of the relativistic stress energy tensor of a fluid

$$T^{\mu\nu} = \rho(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2})\, u^\mu u^\nu - \frac{p}{c^2}\, \eta^{\mu\nu}$$

with

$$u^4 = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad u^i = \frac{U_i}{\sqrt{1 - v^2/c^2}}, \qquad v^2 = \sum_i U_i^2$$

The quantities $\psi_{(i)}$ are not arbitrary but must be chosen so that a set of equations called consistency equations by McVittie must be satisfied. These equations are determined from the requirement that the ten equations (5.7) determine the five quantities ρ, P and U_i. When these are satisfied we find

$$\rho = -\varphi_{,ij}\,\delta^{ij}$$

$$\rho U_i = +\varphi_{,i4} \qquad i,j = 1,2,3 \tag{6.8}$$

$$p = -\varphi_{,44} + \chi$$

where χ is determined by the integrable equations

$$\chi_{,\ell} = \sum_{j=1}^{3} \left(\frac{\varphi_{,4\ell}\,\varphi_{,4j}}{\varphi_{,k\ell}\,\delta^{k\ell}} \right)_{,j} \qquad \ell = 1,2,3 \tag{6.9}$$

We next evaluate the right hand side of equation (4.9) in the classical limit, that is by substituting from equation (6.7) for $T^{\mu\nu}_{(0)}$ and from equations (6.1) and (6.2) with the terms in $1/c^2$ omitted for $h_{\mu\nu}$ and h.

Equation (4.9) then becomes

$$M^{\mu}_{\alpha,\mu} = \frac{k}{2}\left[\varphi_{,\alpha}\,T^{\mu\rho}_{(0)}\,(-2\delta^4_\mu\,\delta^4_\rho + \eta_{\mu\rho}) - 4\varphi_{,\mu}(T^{\mu}_{(0)\alpha} - T^{4\mu}_{(0)}\,\delta^4_\alpha)\right] \tag{6.10}$$

If we set $\alpha = 4$ in this equation we obtain on neglecting terms in $1/c^2$ inside the parentheses,

$$M^{\mu}_{4,\mu} = -\frac{k}{2}\varphi_{,4}\,T^{44}_{(0)} = \frac{k}{2}\varphi_{,4}\,\delta^{ij}\,\varphi_{,ij} \tag{6.11}$$

The first of these equations may be written as

$$\begin{aligned} M^{\mu}_{4,\mu} &= -\frac{k}{2}\left[(\varphi T^{44}_{(0)})_{,4} - \varphi T^{44}_{(0),4}\right] \\ &= -\frac{k}{2}\left[(\varphi T^{44}_{(0)})_{,4} + \varphi T^{4i}_{(0),i}\right] \\ &= -\frac{k}{2}\left[(\varphi T^{44}_{(0)})_{,4} + (\varphi T^{4i}_{(0)})_{,i} - \varphi_{,i}\,T^{4i}_{(0)}\right] \end{aligned} \tag{6.12}$$

since

$$T^{4\nu}_{(0),\nu} = T^{44}_{(0),4} + T^{4i}_{(0),i} = 0$$

The second of equations (6.11) may be written as

$$M^{\mu}_{4,\mu} = \frac{k}{2}\left[(\varphi_{,4}\,\varphi_{,j} - \varphi\varphi_{,4j})_{,i}\,\delta^{ij} - \varphi_{,4i}\,\varphi_{,j}\,\delta^{ij} + (\varphi\varphi_{,4j})_{,i}\,\delta^{ij}\right]$$

$$= \frac{k}{2}\left[(\varphi_{,4}\,\varphi_{,j} - \varphi\varphi_{,4j})_{,i}\,\delta^{ij} - \varphi_{,i}\,T^{4i}_{(0)} + (\varphi\,T^{4i}_{(0)})_{,i}\right] \qquad (6.13)$$

where we have used equations (6.7). Subtracting equation (6.13) from (6.12) we then obtain

$$\varphi_{,i}\,T^{4i}_{(0)} = \frac{1}{2}(\varphi_{,4}\,\varphi_{,j} - \varphi\varphi_{,4j})_{,i}\,\delta^{ij} + (\varphi T^{4i}_{(0)})_{,i} + \frac{1}{2}(\varphi T^{44}_{(0)})_{,4}$$

The scalar φ is related to the Newtonian potential V by the equation

$$V = 4\pi G\varphi\ ,$$

as is evident from the first of equations (6.7). Hence the above equation may be written as

$$\rho\,U_i\,V_{,j}\,\delta^{ij} = \frac{1}{8\pi G}(V_{,4}\,V_{,i} - VV_{,4i})_{,j}\,\delta^{ij} + (\rho V U_i)_{,j}\,\delta^{ij} + \frac{1}{2}(\rho V)_{,4} \qquad (6.14)$$

Equation (6.14) has been derived by H. Bondi from purely classical arguements and has led him to suggest that the vector

$$p_i = \frac{1}{8\pi}(V_{,4}\,V_{,i} - VV_{,4i})$$

is the gravitational analogue of the Poynting vector. That is, it represents the momentum of the gravitational radiation through unit area of a surface exterior to the moving matter.

When equations (6.12) and (6.13) are added we obtain

$$M^{\mu}_{4,\mu} = \frac{k}{4}\left[(\varphi\varphi_{,ij}\,\delta^{ij})_{,4} + (\varphi_{,4}\,\varphi_{,j} - \varphi\varphi_{,4j})_{,i}\,\delta^{ij}\right]$$

or

$$M^{\mu}_{4,\mu} = \frac{1}{c^2}\left[- (\tfrac{1}{2}\rho V)_{,4} + \frac{1}{8\pi G}(V_{,4}V_{,j} - VV_{,4j})_{,i}\,\delta^{ij}\right] \qquad (6.15)$$

This equation relates the four dimensional divergence of the energy-momentum of the material field with the time rate of change of the potential energy of the mass distribution and the divergence of the vector p_i.

We shall compare equation (6.15) to the equation resulting from equation (4.4) by choosing λ_ν as mentioned above. To do this we substitute for $k_{\mu\nu}$ from equations (6.4) with the $1/c^2$ terms omitted into equations (3.10). We then obtain on neglecting $1/c^2$ terms

$$E^{44} = -\frac{k}{2}\left[+ \frac{7}{2}\varphi_{,i}\,\varphi_{,j}\,\delta^{ij} + 4\varphi\varphi_{,ij}\,\delta^{ij}\right]$$

or

$$E^{44} = -\frac{1}{c^2}\left[-4V\rho + \frac{7}{8\pi G}V_{,i}\,V_{,j}\,\delta^{ij}\right] \qquad (6.16)$$

$$E^{4i} = \frac{k}{2}\left[3\varphi_{,4}\,\varphi_{,i} + 4\varphi\varphi_{,i4}\right]$$

or

$$E^{4i} = \frac{1}{c^2(4\pi G)}\left[3V_{,4}\,V_{,i} + 4VV_{,i4}\right] \qquad (6.17)$$

On substituting equations (6.16) and (6.17) into equation (4.4) we obtain

$$M^{\mu}_{4,\mu} = -E^{\mu}_{4,\mu}$$

$$= \frac{1}{c^2}\left[\left(-4V\rho + \frac{7}{8\pi G}V_{,i}\,V_{,j}\,\delta^{ij}\right)_{,4} - \frac{1}{(4\pi G)}(3V_{,4}\,V_{,j} + 4VV_{,4j})_{,i}\,\delta^{ij}\right] \qquad (6.18)$$

Note that equation (6.18) is similar to equation (6.15) in that its right hand member contains products of V and its first and second derivatives. It differs from the right hand member of equation (6.15) in the presence of the term

$$V_{,i}\,V_{,j}\,\delta^{ij}$$